# Predictive Density Combinations with Dynamic Learning for Large Data Sets in Economics and Finance

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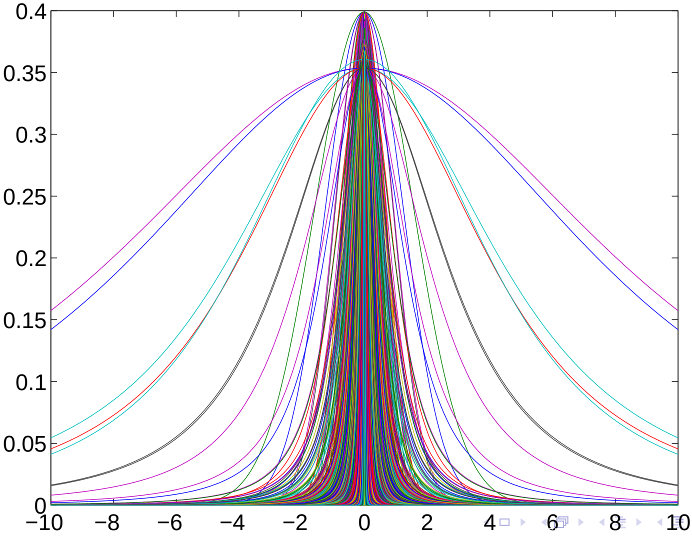
# Motivation: Average 7424 Density Forecasts of 1856 Stock Returns



# Motivation

- **Problem and Practice** Many agencies handle this averaging informally. Our aim: Give this a **Bayesian probabilistic foundation** in order to evaluate practical issues like: Probabilities of (extreme) events : Recession probability; Turning point probability; Probabilistic warnings about defaults; Value-at-Risk etc etc.
- **Fast growth in Big Data** give more accurate measures. Analogy with weather forecasting using many satellite pictures. But multimodality, skewness etc in economics.
- **Parallel Computing: New Hardware and Software** give openings to solve complex problems. **Machine learning with several hidden layers using neural networks have a direct connection with filtering methods in nonlinear time series models**.

# Three Theoretical Contributions

- **Flexible Bayesian Combination Model is extension of Mixture of Experts** Model, (Jordan and Jacobs, 2010 and many others) allowing for cross-section and time dependent **Bayesian weight learning** and **diagnostic learning about model incompleteness**.
- **Dimension reduction in three steps**
    - **Sequential clustering** of large set of predictions with learning weights.
    - **Dynamic learning** of weights on a **simplex of reduced dimension**. (Time series behaviour on a bounded domain).
    - **From small simplex to large simplex** using **class-preserving property** of the logistic-normal distribution.
- **Model Representation and Efficient Computation**
    - Model is a **nonlinear State Space Model**
    - **K-means clustering algorithm** and **Nonlinear Sequential Filtering**. Filtering is parallelised and directly connected to hidden layers in machine learning with neural network weights

# Review of the literature

- Averaging for forecast accuracy (Barnard (1963), Bates and Granger (1969)).
- Parameter and model uncertainties importance (BMA, Roberts (1965)).
- Probabilistic predictions provide larger set of information (Wallis).
- Correlations between forecasts and weights (Garratt, Mitchell and Vahey (2012)).
- Model performances differs over regions of interest/quantiles (mixture of predictives; generalized LOP: Fawcett, Kapetanios, Mitchell and Price, 2014; subsets of interest: Pelenis, 2014).
- Model performance varies over time possibly with persistence (Diebold and Pauly (1987), Guidolin and Timmermann (2009), Hoogerheide et al. (2010), Gneiting and Raftery (2007); Billio et al. (2013); Del Negro, Hasegawa and Schorfheide, 2015).
- Models might perform differently for multiple variables of interest (specific weight for each series, univariate models).
- Model set is possible incomplete (Geweke (2009), Geweke and Amisano (2010), Waggoner and Zha (2010)).
- GPU computing improve speed, see Casarin, et al. (2014), Dziubinski and Grassi(2013), Geweke and Durham (2012) and many others.

# Review earlier density combination

- Billio, Casarin, Ravazzolo and Van Dijk (2013, JE) propose a **distributional state-space representation** of the predictive densities and the combination scheme
- Casarin, Grassi, Ravazzolo and Van Dijk, (2015, JSS) created **a MATLAB toolbox DECO** for estimating the density combination scheme of BCRVD (2013).
- Extensions to **Nowcasting** in Aastveit, Ravazzolo and Van Dijk (2018, JBES); **Forecasting Combinations and Portfolio Combinations** in Basturk, Borowska, Grassi, Hoogerheide and Van Dijk (2018). In this paper: **Large Data**
- **Bayesian Foundational** paper McAllinn and West (2018)
- Background: **Evolution of Density Combinations in Economics**, Aastveit, Mitchell, Ravazzolo and Van Dijk(2018)
- software on `http://www.francescoravazzolo.com/pages/DeCo.html`

# Mixture representation extends mixture of experts model

Basic Idea for one economic variable $y_t$

- **Basic practice** $\sum_{i=1}^{n} w_{it}\tilde{y}_{it}$.
- **Formally: conditional predictive probability** of $y_t$, given $\tilde{\mathbf{y}}_t = (\tilde{y}_{1t}, \ldots, \tilde{y}_{nt})'$, is **discrete mixture of conditional predictive probabilities** of $y_t$ given $\tilde{y}_{it}$, $i = 1, \ldots, n$ from $n$ individual models with weights, $w_{it}$, $i = 1, \ldots, n$. **Fundamental density combination**

$$f(y_t|\tilde{\mathbf{y}}_t) = \sum_{i=1}^{n} w_{it} f(y_t|\tilde{y}_{it})$$

- **Discrete/continuous mixture representation**
  Under standard regularity conditions the **marginal predictive density of $y_t$** has the following discrete/continuous representation:

$$f(y_t|I) = \sum_{i=1}^{n} w_{it} \int_{\mathbb{R}} f(y_t|\tilde{y}_{it}) f(\tilde{y}_{it}|I_i) d\tilde{y}_{it}$$

# Mixture representation with learning about model incompleteness and dynamic weight behaviour

- We specify a Gaussian combination density $f(y_t|\tilde{y}_{it}) = \mathcal{N}\left(y_t|\tilde{y}_{it}, \sigma_t^2\right)$ with stochastic volatility, $\sigma_t^2$ **that determines overall uncertainty and indicates model incompleteness.** If $\sigma_t^2$ tends to zero, one obtains the static mixture of experts model from Jacobs and Jordan and Geweke and Keane. Also choice of the model set is important.
  This leads to **diagnostic learning.**
- Dynamic weights are **periodically updated with Bayesian learning**. (Here simple random walk).

# Number of predictions is large, say $n$, reduce this to a small set, say $m$. Dimension reduction in three steps

- **Step 1: Sequential clustering of predictions to $m$ clusters** with learning following some features of the predictive densities. Grouping can change over time. Weights driven by a model-specific predictive performance measure like log score (or equal weights).
- **Step2: Unrestricted weights $v_t$ of clusters** follow a $m$-variate normal random walk process or more involved **Bayesian learning** and **are mapped to low dimensional simplex to have probabilistic interpretation of model weights, $z_t$**
- **Step 3: Map weights $z_t$ from small $m-1$ simplex back to large $n-1$ simplex to $w_t$ using the class-preserving property of the logistic-normal distribution** from **Aitchinson's geometry of the simplex** so that typical feature of model probability weight is preserved.

# Summary of Data Driven Density Combinations for Large Datasets and connection with Machine Learning

| | Densities for Large Datasets (7400 densities) | | | |
|---|---|---|---|---|
| | ↘ | ↓ | ↓ | ↙ |
| | Sequential clustering step | | | |
| | ↓ | ↓ | ↓ | ↓ |
| Hidden layers with unrestricted cluster weights $v$ and scaled probabilistic weights $z$ | Cluster 1 | Cluster 2 | Cluster 3 | Cluster 4 |
| | ↓ | ↓ | ↓ | ↓ |
| | $z_1 = g(v_1)$ | $z_2 = g(v_2)$ | $z_3 = g(v_3)$ | $z_4 = g(v_4)$ |
| | ↘ | ↓ | ↓ | ↙ |
| | Updating Clustered Density Combinations | | | |
| | | ↓ | ↓ | |
| | Original series with weights $w$ | | | |
| | ↙ | ↓ | ↓ | ↘ |
| | Densities with weights for all series | | | |

**Results**

- **Hidden layer weights integrated using filtering. Also neural networks from machine learning.**
- Mixtures on **large space have same properties as reduced space.**

# Representation result: (I) Nonlinear SSM as finite mixture with large sets of members and transition function as logistic-normal weight

Model: **marginal** predictive density of $y_t$ with normal combination density, time-varying volatility for model incompleteness; dynamic learning of clustered weights

$$f(y_t|I) = \sum_{i=1}^{n} w_{it} \int_{\mathbb{R}} \mathcal{N}\left(y_t|\tilde{y}_{it}, \sigma_t^2\right) f(\tilde{y}_{it}|I_i) d\tilde{y}_{it}$$

## Nonlinear State Space Model

$$\mathbf{y}_t \sim \sum_{i=1}^{n} w_{it} \mathcal{N}\left(\tilde{y}_{it}, \sigma_t^2\right) \quad (1)$$

$$\tilde{\mathbf{w}}_t \sim \mathcal{L}_{n-1}\left(\tilde{B}_t D_m \mathbf{v}_{t-1}, \tilde{B}_t D_m \Sigma D_m' \tilde{B}_t'\right), \quad (2)$$

$\tilde{\mathbf{w}}_t = (w_{1,t}, \ldots, w_{n-1,t})'$ and $w_{n,t} = 1 - \tilde{\mathbf{w}}_t' \boldsymbol{\iota}_{n-1}$ and $\tilde{B}_t$ contains weights.

# Representation result: (II) Under regular conditions the Model is Generalized Linear Model with Nonlinear Local Level Model

## Corollary

*Let* $\mathbf{s}_t$ *be an allocation vector, with* $\mathbf{s}_t \sim \mathcal{M}_n(1, \mathbf{w}_t)$*, where* $\mathcal{M}_n(1, \mathbf{w}_t)$ *denotes the multinomial distribution, and let* $\sigma_t$ *be a time-varying variance. Then, the state space model given in the Proposition can be written as*

$$\mathbf{y}_t = \tilde{\mathbf{y}}_t' \mathbf{s}_t + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(\mathbf{0}, \sigma_t) \tag{3}$$

$$\mathbf{s}_{i,t} = \begin{cases} 1 & \textit{with probability } w_{i,t} \\ 0 & \textit{otherwise} \end{cases} \tag{4}$$

$$\mathbf{w}_t = \boldsymbol{\phi}_{\boldsymbol{B}}(\mathbf{z}_t) \tag{5}$$

$$\mathbf{z}_t = \mathbf{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \mathcal{L}_{m-1}\left(\mathbf{0}, D_m \Sigma D_m'\right) \tag{6}$$

*where* $\boldsymbol{\phi}_{\boldsymbol{B}}(\mathbf{z}_t)$ *is a nonlinear logistic transformation*

# Two Numerical approximations (prior is diffuse but proper)

- (1) Clustering-based mapping of the predictors requires the solution of an optimization problem which is not available in analytic form;
- (2) Analytic solution of the optimal filtering problem is generally not known;
- Apply sequential numerical approximation algorithms which, at time $t$ iterate over the following steps:

1. **Parallel k-means clustering on GPU,** based on features $\boldsymbol{\psi}_{it}$ where the centroids are updated dynamically as follows:

$$\mathbf{c}_{jt+1} = \mathbf{c}_{jt} + \lambda_t(\mathbf{m}_{jt+1} - \mathbf{c}_{jt})$$

where

$$\mathbf{m}_{jt+1} = \frac{1}{n_{jt+1}} \sum_{i \in N_{jt+1}} \boldsymbol{\psi}_{it}$$

and $\lambda_t \in [0, 1]$. This implies a sequential clustering with forgetting driven by the processing of the blocks of observations.

2. **Sequential (over time dimension) Monte Carlo approximation** of the nonlinear SSM using GPU (more later)

# Two Groups of Empirical Contributions

- **Empirical 1**: Financial Time Series: Data are 1856 individual stock prices, quoted in NYSE and NASDAQ. Predict Features of a Replication of S&P500 index.
  - **Better accuracy than standard models about many density features: means, volatilities and, in particular, tails.**
  - **Learning about** time behaviour of clusters of stocks. **Joint** dependence over time among weights. Signal of model incompleteness from diagnostic learning.
  - Prediction of the **statistical and economic accuracy of a tail estimates: event like Value-at-Risk**.

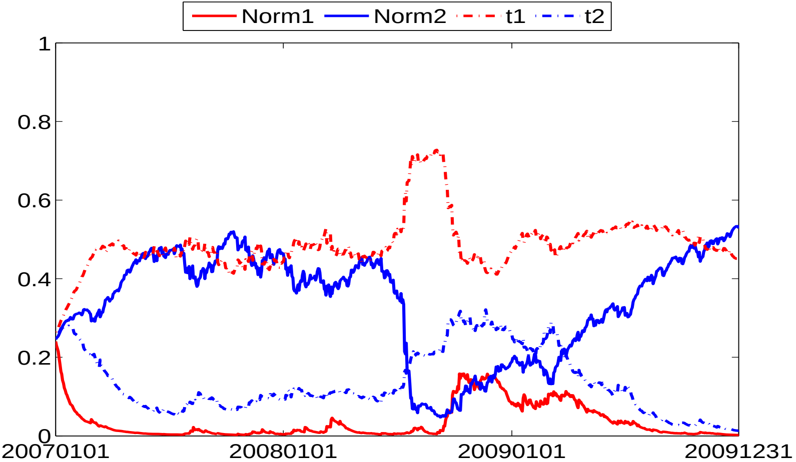
# Empirical 1: Financial time series: Models and Estimation

- Data: 1856 individual stock daily prices quoted in the NYSE and NASDAQ from Datastream over the sample March, 18, 2002 to December, 31, 2009.
- Models and Estimation: Estimate a Normal GARCH(1,1) model and a $t$-GARCH(1,1) model via posterior mode using rolling samples of 1250 trading days (about five years) for each stock return:

$$
\begin{aligned}
y_{it} &= c_i + \kappa_{it}\varepsilon_{it} \\
\kappa_{it}^2 &= \theta_{i0} + \theta_{i1}\varepsilon_{i,t-1}^2 + \theta_2\kappa_{i,t-1}^2
\end{aligned}
$$

  where $y_{i,t}$ is the log return of stock $i$ at day $t$, $\varepsilon_{it} \sim \mathcal{N}(0,1)$ and $\varepsilon_{it} \sim \mathcal{T}(\nu_i)$ for the Normal and Student-$t$ cases, respectively.
- Produce 784 one day ahead density forecasts for the period January 1, 2007 to December 31, 2009.

# Empirical 1: Dynamic behaviour of weights of 4 clusters




Normal GARCH(1,1) with **low (cluster 1) vol.** and **high (cluster 2) vol.**; *t*-GARCH(1,1) with **low (cluster 3)** and **high (cluster 4) d.o.f.**. Three subperiods: Weights adjust to time instability: Lehman Brothers Default

# Empirical 1: Time movement of joint dependence of weights using Canonical Correlations

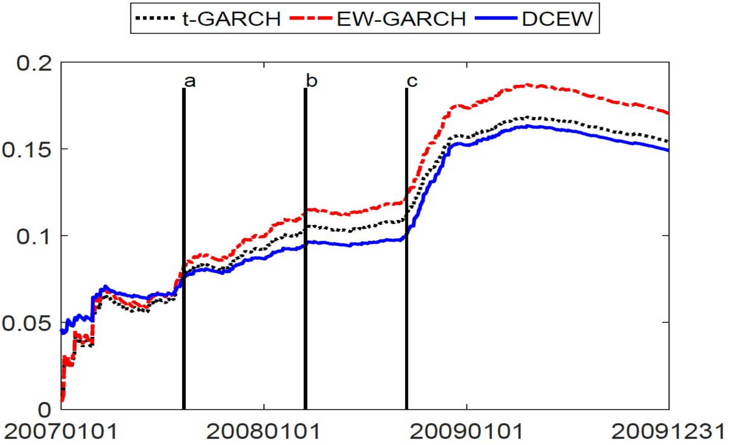

Each cluster versus all others. Change in cluster 2 and 3 after Lehman

# Empirical 1: Forecasting Features of Densities: Means; Log Scores: Tail probabilities and Risk

| | RMSPE | LS | CRPS | avQS-T | avQS-L | Violation |
|---|---|---|---|---|---|---|
| WN | 1.852 | -9.045 | 1.017 | 0.429 | 0.425 | 3.57% |
| Normal GARCH | 1.852 | -4.164$^{**}$ | 0.956$^{**}$ | 0.139$^{**}$ | 0.195$^{**}$ | 2.93% |
| *t*-GARCH | 1.852 | -2.738$^{**}$ | 0.937$^{**}$ | 0.118$^{**}$ | 0.154$^{**}$ | 2.55% |
| GJR-GARCH | 1.852 | -4.068$^{**}$ | 0.955$^{**}$ | 0.125$^{**}$ | 0.158$^{**}$ | 2.75% |
| EW-GARCH | 1.853 | -3.145$^{**}$ | 1.018 | 0.144$^{**}$ | 0.171$^{**}$ | 2.80 |
| DCEW | **1.812**$^{**}$ | **2.249**$^{**}$ | **0.911**$^{**}$ | **0.114**$^{**}$ | **0.149**$^{**}$ | 0.90% |
| DCEW-SV | 1.816$^{**}$ | 2.206$^{**}$ | **0.913**$^{**}$ | 0.114$^{**}$ | 0.149$^{**}$ | **1.02%** |

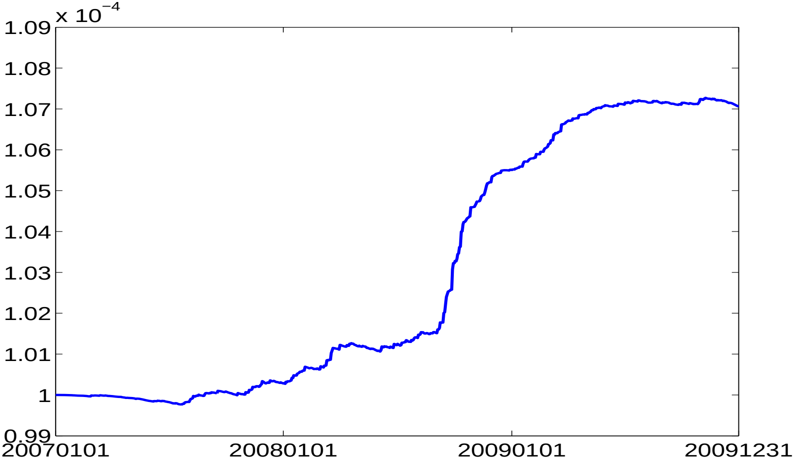
Table: Forecasting results for next day S&P500 log returns.
avQS-T and avQS-L: average tails (T) and average left tail (L) scores described in Gneiting and Raftery (2007).
Violation: percentage of times the realization exceeds the 1% Value at Risk (VaR).

# Empirical 1: Cumulative left quantile score for 3 cases




Timeline legend: a - 8/9/2007, BNP Paribas redemptions on three investment funds; b - 3/17/2008, collapse of Bear Stearns; c - 9/15/2008, Lehman bankruptcy. Stock market stress increases gap between models. Accuracy drops after Lehman default

# Empirical 1: Model incompleteness measured by $\sigma_t$




Variance increases in DCEW-SV scheme in September 8, 2008. Signal for including possibly models with jumps in volatility

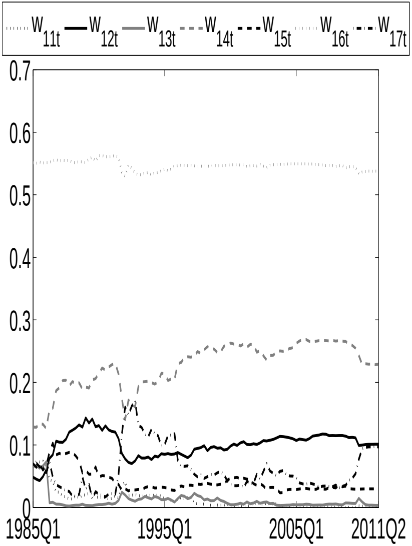
# Empirical 2: Stock and Watson (2005) dataset. Models and Estimation GDP, GDP deflator, 3-month Interest rates, Employment

- **Empirical 2**: Macroeconomic Time Series. Extend Stock and Watson (2002,2005) as follows:
  - **Joint prediction model** for the group of variables Real GDP, GDP deflator, Treasury Bill rate, Employment instead of a single variable 'beats' all alternatives considered.
  - **Learning about dynamic behaviour of 5-7 sectors**. Factor-model combination shows rise and decline of sectors. Dominant factor?
  - **Accurate probability of turning point and of recession measured over time**

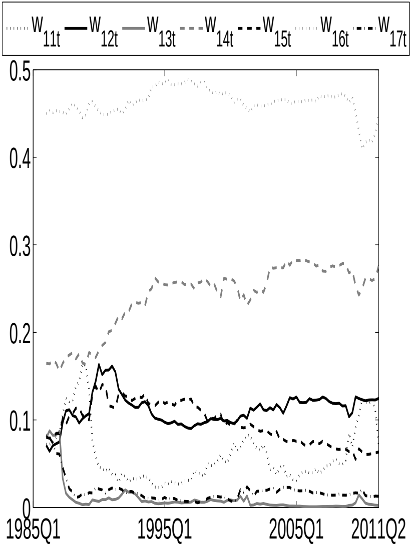
# Empirical 2: Stock and Watson (2005) dataset. Models and Estimation GDP, GDP deflator, 3-month Interest rates, Employment.

- Stock and Watson (2005) dataset: 142 series, standardized and sampled at a quarterly frequency from 1959Q1 to 2011Q2.
- Bayesian estimation of AR(1) process for all series and group them using residual variance and the persistence parameters.Identify $m = 5, 7$ clusters.
- Combination: univariate and multivariate combinations, 5 and 7 clusters, equal and log score weights.
- Produce from 1 to 5-step ahead recursive AR(1) forecasts for the out-of-sample: 1985Q1-2011Q2.
- Evaluation criteria: MSPE, CRPS and Log Score.

# Empirical 2: Dynamic behaviour of cluster weights of GDP growth: dominant is 6 (Exports, Imports, GDP deflator)

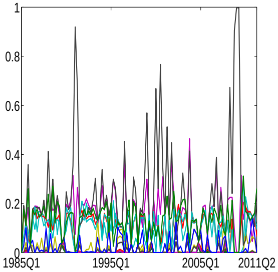



1-step ahed

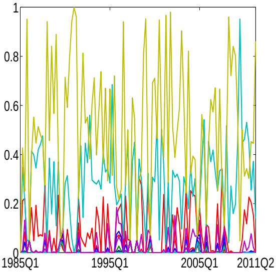



5-step ahead

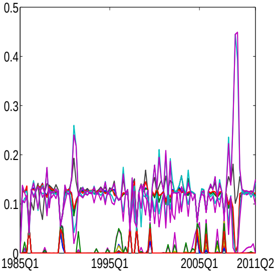
# Empirical 2: Dynamic behaviour of model weights $w$ of GDP growth, 1-step ahead




cluster 1




cluster 2




cluster 3

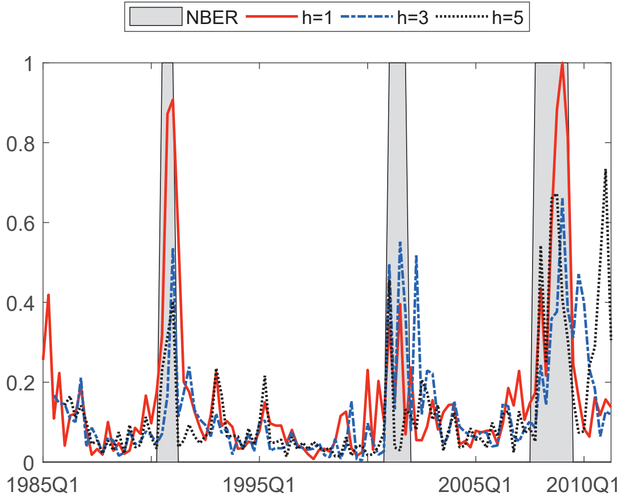
# Empirical 2: 8 Competitive models for Forecasting

- UDC versus MDC (univariate versus multivariate combination)
- EW versus LS (equal weights versus recursive log score weights);
- 5 versus 7 (5 cluster versus 7 clusters);
- 8 models: UDCEW5, MDCEW5, UDCLS5, UDCLS7, MDCLS5, UDCEW7, MDCEW7, UDCLS7, MDCLS7

# Empirical 2: GDP Forecasting Results

| | h=1 | | | h=5 | | |
|---|---|---|---|---|---|---|
| | MSPE | LS | CRPS | MSPE | LS | CRPS |
| AR | 0.647 | -1.002 | 0.492 | 0.682 | -1.009 | 0.506 |
| BDFM | 0.649 | -1.091 | 0.382** | 0.655 | -1.099 | 0.388** |
| UDCEW5 | 0.644 | -0.869 | 0.333** | 0.658* | -0.912 | 0.343** |
| MDCEW5 | 0.63 | -0.928 | 0.326** | 0.636* | -0.844 | 0.324** |
| UDCLS5 | 0.773 | -1.306 | 0.464 | 0.715 | -1.38 | 0.481 |
| MDCLS5 | 0.725 | -1.145 | 0.505 | 0.557* | -1.005 | 0.358** |
| UDCEW7 | 0.649 | -0.875 | 0.334** | 0.657* | -0.891 | 0.338** |
| MDCEW7 | 0.642 | -0.979 | 0.334** | 0.654* | -1.009 | 0.342** |
| UDCLS7 | 0.646 | -0.868* | 0.332** | 0.657* | -0.914 | 0.342** |
| MDCLS7 | **0.596*** | **-0.586**** | **0.275**** | **0.610**** | **-0.634**** | **0.286**** |

# Empirical 2: Probabilities of negative quarterly growth




One quarter ahead, three quarters ahead and five quarter ahead probabilities over time of negative quarterly growth given by the the combination approach and ex-post NBER

# Conclusions

- **Time-varying combinations** of large sets of predictive densities based on **sequential clustering** analysis can deal with **big data**.
- Combination weights are driven by **cluster-specific latent processes much smaller** than the number of available predictors.
- **The proposed model is a nonlinear SSM with finite mixtures and dynamic logistic-normal weights. Density combination evaluated by nonlinear sequential filtering. Interesting connections with machine learning with neural nets**
- Forecasting financial time series: Three improvements: **Forecast accuracy of moments; dynamic learning about clusters; efficient estimates of risk**
- Substantial gains in point and density forecasting of **joint** set of US real GDP, GDP deflator, Treasury Bill returns and employment growth based on log score learning. **Rise and decline of sectors; probabilities of recession.**
- Parallelisation:Handling big data sets with GPU can be easy and fast.

# Topics for Further Research

- **Modelling and estimation**. Use **more information** on clustering and explore diagnostic analysis about model incompleteness and richer model set.
- More on **efficiency of filter** methods: New Filter obtained using the mixture of student density approximation constructed using an EM weighted importance sampling approach, HOVD(2012, JE), labeled M-Filter. Forthcoming Journal of Econometrics.
- **Forecasting and Policy**. More on Nowcasting and Multiperiod out-of-sample forecasting. Applications to **Policy Issues**: In the field of Finance using Decision Models. Current research on Bayesian Dynamic Modelling and Time-varying combinations of Equity Momentum Strategies using US industrial portfolios 1929-2015. **Time-varying combinations jointly for models and policies: mixture of mixtures**. Challenge to do this for macro-models.
- **More efficient parallel computing.**